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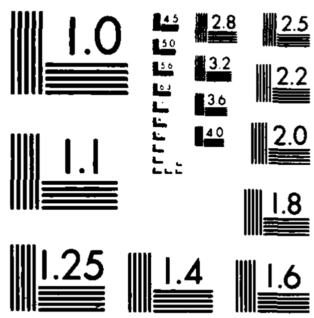

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NAVAL POSTGRADUATE SCHOOL

Monterey, California

INSPECTION POLICIES FOR STAND-BY SYSTEMS

by

L. C. Thomas
P. A. Jacobs
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An auxiliary service unit is normally idle, or in cold standby. If a demand for the unit's service occurs, the unit must be available to satisfy it, or else "catastrophe" occurs. Policies for periodic inspection and maintenance of such a unit are derived in this paper that maximize the expected time until a catastrophe occurs. The policies recognize that inspection, maintenance, and repair periods are of non-zero duration, during which the unit is vulnerable. They also account for the possibility of hazardous inspection

that may damage the unit, and various forms of imperfect repair.

Important examples occur in the nuclear power industry: a unit may be a pump, or emergency diesel generator, and a demand may be caused by an initiating event such as pipe break or loss of off site power; "catastrophe" equates to loss-of coolant accident or melt down. Other examples occur in the military, and in emergency services to hospitals.

Inspection Policies for Stand-by Systems

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Key words: Reliability, availability, maintenance, time to failure, inspection, Markov decision process, nuclear safety, standby redundancy.

1. INTRODUCTION

It is common practice to improve the reliability of a system by installing cold standby units, which are only brought into operation when a standard operating system fails. In particular, diesel generators in cold standby may be used to scram a reactor in case of a coolant pipe breaking or some other failure in a nuclear power plant. Other examples occur in hospital power supplies and military hardware. If such a standby system fails to operate when it is required, then the consequences could be catastrophic. The times when there is a need for the standby unit are called initiating events. If the standby system is in a failed state, when an initiating event occurs, then a catastrophic event is said to occur.

It is necessary to inspect and maintain the standby system from time to time. If inspection reveals it to be in an unsatisfactory state, repairs are made. The idea is that the standby unit can go down even when it is not operating and this will cause it to fail to operate the next time it is needed.

The following policy has been proposed for the inspection of diesel generators in a reactor. After a generator is found to be down on inspection and is repaired, it undergoes K inspections at short intervals of time. If it is found to be up at each of these short inspections, then it is inspected at long intervals thereafter until it is found to be down. Whenever a generator is found to be down and is repaired, inspections start with the K short inspection intervals again. This type of inspection policy reflects the idea that after the system is repaired it

should be inspected more often for awhile to ensure it was repaired correctly. In Section 2 we present a model for this inspection policy and derive an expression for the expected time to a catastrophic event.

In Sections 3 through 5 we will use various Markov decision and renewal theoretic formulations of the problem to investigate the forms of the <u>optimal</u> inspection policies which maximize the expected time until a catastrophic event occurs. This will show us how certain assumptions about inspection and repair of the standby system affect the form of the inspection policy.

Almost all the previous work on inspecting a single standby unit uses a cost criterion. Barlow and Proschan [2] described the basic average cost per unit time model with accurate instantaneous inspection and faultless repair, while Luss and Kander [9] allowed for non-zero inspection times. Wattanapanom and Shaw [20] studied the problem when inspection is hazardous, so that it is possible for the inspection to cause the unit to fail. Nakagawa [11] looked at the probability that at an initiating event the standby system will work, while Butler [3] maximized the expected lifetime of the standby unit, but did not allow repairs. His model allowed the standby unit to be in more than one 'up' state, which are distinguishable only upon inspection. This connects with the work on partially observable Markov decision processes [1,10,16], and in particular the problem of optimal inspection and repair of a deteriorating process with imperfect information introduced by Ross [13] and generalized by White [21], Rosenfield [12], Luss [8], Sengupta [15], Suzuki [17], and Wong

[19]. In these papers, a system can be in more than one state, but which one is known only imperfectly or upon inspection.

Our models of the inspection and repair of the standby system allow for non-zero inspection-maintenance times and non-zero repair periods, but we ignore the time the unit is in use. The idea is that during inspection-maintenance and repair the unit can not react to an initiating event and so these are critical times for the system, whereas we make the assumption that the time the standby system is actually in use is so small it can be neglected. We also allow for imperfect repair and hazardous inspection, so that even if the unit is up on inspection, it might be down immediately after. Thus we explicitly represent possible mistakes in inspection, and allow for incorrectly identifying the unit as working when in fact it was down. Another model considered allows the unit to be in one of two 'up' states, which are indistinguishable on inspection, but have different failure rates. This is intended to incorporate the idea that a repair might put right the superficial cause of the unit's failure, but not deal with the underlying problem, which will recur.

In Section 3, we introduce our basic discrete time models where the unit can only be either 'up' or 'down'. The times between initiating events are assumed to have a geometric distribution. We describe the case where successfully dealt with initiating events are recorded as showing the unit was working at that time. By modelling this as a Markov decision process we can find the form of the optimal inspection policy to maximize expected time to a catastrophic event. We compare this with the

case where we ignore any information from successfully dealt with initiating events. We also look at the expected times until a catastrophic event under different policies, and optimize the probability that the system will last at least a fixed number of time periods. Section 4 describes the equivalent continuous time model and shows how the discrete time results are replicated if the lifetime of the unit is exponential and the initiating events occur according to a Poisson process. We also investigate the optimal inspection policy for general lifetime distributions. Section 5 generalizes the discrete time model to allow the unit to be in two 'up' states. In certain cases the optimal inspection policy for this model has quite short inspection periods immediately after a repair, which then lengthen as further inspections suggest the system is in the "better" up state.

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2. CONTINUOUS TIME MODEL WITH TWO-UP STATES AND SHORT-LONG INSPECTION POLICY

Assume the system can be in one of two up-states $j = 1,2$ until it fails. The two up-states are indistinguishable upon inspection. After a repair the system goes to up-state j with probability π_j and remains there until it fails. After a repair the conditional distribution of the time to failure given it is in up-state j is G_j, independent of the past.

After a repair the system is inspected and maintained at K short intervals of length S. If the system is found to be up at each of the K short inspection intervals, then future inspections occur at long intervals of length $L > S$. If the system is found to be down upon inspection, it is repaired and then inspected at K short inspection intervals again before the long inspection intervals begin. If the system is found to be up upon inspection, routine maintenance is performed. Given the system is in up-state j, the conditional distribution of the time to failure after an inspection is F_j, independent of the past. Some reasonable and tractable examples of distributions F_j and G_j are the exponential, and the exponential with a probability atom at the origin reflecting hazardous inspection or faulty repair.

Inspection-maintenance takes M units of time and repair takes R units of time. Initiating events occur according to a Poisson process with rate ν. The system is unable to respond to an initiating event during inspection-maintenance or repair. A catastrophic event is said to occur if an initiating event

occurs when the system has failed or is being inspected, maintained, or repaired. Let T denote the time of the first catastrophic event. We will derive an expression for the expected value of T.

Let $f(j,k) = E_{j,k}[T]$ denote the expected time to the first catastrophic event given $k = 0,1,\ldots,K$ short inspection periods have already successfully taken place and the system is in up-state j. Let $f(j,\ell) = E_{j,\ell}[T]$ denote the expected time to first catastrophic event given a successful inspection has just taken place, the next inspection period is long, and the system is in up-state j.

A probabilistic argument gives the following system of equations; $(\overline{F}_j(S) = 1 - F_j(S))$.

$$\begin{aligned} f(j,0) = {} & \overline{G}_j(S)e^{-\nu M}\{S+M+f(j,1)\} \\ & + \overline{G}_j(S) \int_0^S (S+u)\nu e^{-\nu u}\, du \\ & + \int_0^S G_j(du) \int_0^{S-u+R} (u+z)\nu e^{-\nu z}\, dz \\ & + \int_0^S G_j(du)e^{-\nu(S-u+R)}[S+R+\sum_{j=1}^{2} \pi_j f(j,0)]\ ; \end{aligned} \tag{2.1}$$

for $1 \le k \le K-1$,

$$f(j,k) = \bar{F}_j(S)e^{-\nu M}\{S+M+f(j,k+1)\} \tag{2.2}$$

$$+ \bar{F}_j(S) \int_0^M (S+u)\nu e^{-\nu u}\, du$$

$$+ \int_0^S F_j(du) \int_0^S (u+z)\nu e^{-\nu z}\, dz$$

$$+ \int_0^S F_j(du)e^{-\nu(S-u+R)}[S+R+\sum_{j=1}^{2} \pi_j f(j,0)] ,$$

where $f(j,K) = f(j,\ell)$;

$$f(j,\ell) = \bar{F}_j(L)e^{-\nu M}[L+M+f(j,\ell)] \tag{2.3}$$

$$+ \bar{F}_j(L) \int_0^M (L+u)\nu e^{-\nu u}\, du$$

$$+ \int_0^L F_j(du) \int_0^{L-u+R} (u+z)\nu e^{-\nu z}\, dz$$

$$+ \int_0^L F_j(du)e^{-\nu(L-u+R)}[L+R+\sum_{j=1}^{2} \pi_j f(j,0)] .$$

After some simplification, equations (2.1)-(2.3) become

$$f(j,0) = a_0(j,S) + p_0(j,S)f(j,1) + c_0(j,S)\pi f(0) ; \tag{2.4}$$

for $1 \le k \le K-1$

$$f(j,k) = a(j,S) + p(j,S)f(j,k+1) + c(j,S)\pi f(0) \; ; \tag{2.5}$$

$$f(j,\ell) = a(j,L) + p(j,L)f(j,\ell) + c(j,L)\pi f(0) \tag{2.6}$$

where

$$\pi f(0) = \sum_{j=1}^{2} \pi_j f(j,0) \; ;$$

$$p_0(j,S) = \overline{G}_j(S)e^{-\nu M} \; ; \tag{2.7}$$

$$c_0(j,S) = e^{-\nu(S+R)} \int_0^S e^{\nu u} \, G_j(du) \; ; \tag{2.8}$$

$$a_0(j,S) = \frac{1}{\nu}\,[1 - p_0(j,0) - c_0(j,0)] \tag{2.9}$$

$$+ \overline{G}_j(S)S + \int_0^S uG_j(du) \; ;$$

$$p(j,t) = \overline{F}_j(t)e^{-\nu M} \; ; \tag{2.10}$$

$$c(j,t) = e^{-\nu(t+R)} \int_0^t e^{\nu u} F_j(du) \; ; \tag{2.11}$$

$$a(j,t) = \frac{1}{\nu}\,[1 - p(j,t) - c(j,t)] \tag{2.12}$$

$$+ \overline{F}_j(t)t + \int_0^t u \, F_j(du) \; . \tag{2.12}$$

In the special case in which F_j has an exponential distribution with an atom at the origin,

$$F_j(t) = \begin{cases} 0 & \text{if } t < 0 , \\ (1-\alpha_j) + \alpha_j[1-e^{-\alpha_j t}] & \text{if } t \geq 0 \end{cases}$$

then

$$p(j,t) = \alpha_j e^{-\delta_j t} e^{-\nu M} \tag{2.13}$$

$$c(j,t) = e^{-\nu(t+R)}(1-a_j) + a_j \frac{\delta_j}{\delta_j-\nu}[1-e^{-(\delta_j-\nu)t}] \tag{2.14}$$

$$a(j,t) = \frac{1}{\nu}[1-p(j,t) - c(j,t)] \tag{2.15}$$

$$+ \alpha_j \frac{1}{\delta_j}[1-e^{-\delta_j t}]$$

Solving equations (2.4)-(2.6) recursively leads to the following expression for the expected time to the first catastrophic event given the system has just been repaired

$$\pi f(0) = \frac{\text{NUM}}{\text{DEN}} \tag{2.16}$$

where

$$\text{NUM} = \sum_{j=1}^{2} \pi_j [a_0(j,S) + p_0(j,S) g_N(K-1)] \tag{2.17}$$

where

$$g_N(K-1) = \frac{[1-p(j,S)^{K-1}]}{1-p(j,S)} a(j,S) + p(j,S)^{K-1} \frac{a(j,L)}{1-p(j,L)} ; \tag{2.18}$$

$$\text{DEN} = \sum_{j=1}^{2} \pi_j [1-\{c_0(j,S) + p_0(j,S) g_D(K-1)\}] \qquad (2.19)$$

$$g_D(K-1) = \frac{[1-p(j,S)^{K-1}]}{1-p(j,S)} c(j,S) + p(j,S)^{K-1} \frac{c(j,L)}{1-p(j,L)}. \qquad (2.20)$$

EXAMPLE. The rate of initiating events is $\nu = 0.1$ per week. $\pi_1 = 0.9 = 1 - \pi_2$. The length of an inspection-maintenance period M is $\frac{0.25}{7}$ weeks. A repair period, R, is $\frac{0.5}{7}$ weeks.

$$F_j(t) = \begin{cases} 0 & \text{if} \quad t < 0 , \\ (1-OKI) + OKI[1-e^{-\delta_j t}] & \text{if} \quad t \geq 0 \end{cases}$$

$$G_j(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ (1-OKR) + OKR[1-e^{-\delta_j t}] & \text{if} \quad t \geq 0 \end{cases}$$

Assume $\delta_1 = \frac{9}{258}$ per week, $\delta_2 = \frac{1}{2}$ per week. Note that after a repair the conditional expected time to system failure given the system is up is $\pi_1 \frac{1}{\delta_1} + \pi_2 \frac{1}{\delta_2} = 26$ weeks. Thus, if after a repair, no inspections are done, then the expected time to a catastrophic event is (OKR)(26) + $\frac{1}{\nu}$ = (OKR)(26) + 10 weeks.

An exploratory numerical study was conducted of the best values of S, L, and K for various values of OKI, OKR. We restricted our attention to the case in which inter-inspection periods are in integer numbers of weeks. Equations (2.16)-(2.20) were evaluated numerically for various parameter values. Some results are summarized in Table 1.

Table 1

OKI	OKR	Best S	Best K	Best L	Best $\pi f(0)$	Expected Time if no inspections
0.9	0.9	1	2	2	69.24	33.4
0.5	0.5	∞	-	-	23	23
0.5	0.9	9	1	1	38.39	33.4
0.9	0.5	1	1	3 or 4	50.63	23

If the quality of the repair is better than the quality of inspection (OKR > OKI) then it appears to be better not to inspect often initially after a repair but then to inspect more often as time goes on. If OKI > OKR then it appears to be better to inspect soon after a repair and if the system is up at inspection not to inspect for a longer period of time thereafter. If both repair and inspection are of poor quality then it appears to be better not to do anything. Note that the expected time to a catastrophe seems to be more sensitive to OKI than to OKR.

In the remainder of the paper we will study optimal inspection policies.

3. DISCRETE TIME, ONE-UP-STATE, MARKOV DECISION PROCESS MODELS

MODEL 1

In the first model, the standby unit can either be 'up' or 'down', when it is not in operation; and, if n basic time periods e.g., days, have elapsed since the unit was installed, s_n is the probability that it will be 'up' at the next time period given that it is 'up' in this (the n^{th}) time period. Once the unit goes 'down' it remains 'down' until either it is successfully repaired or else a catastrophic initiating event occurs. Each time period, the operator can inspect the unit, repair it, or do nothing. If the inspection finds the unit is 'up', no repairs are made, but there is a probability (1-i) that the inspection was actually hazardous or damaging, and so the unit is 'down' immediately after inspection. An inspection which finds the unit up takes M periods, where M need not be integer; during this period the unit cannot respond to an initiating event. If, on inspection, the unit is found in the down state, a repair is attempted, which with probability r will return the unit to the 'up' state and with probability (1-r) leaves it in the down state; this takes a total time of R periods to perform ($R \geq M$); again the unit cannot respond to an initiating event during this period. If the operator decides on a repair without inspection, the unit is again out of operation for R periods and has probability r of being in the 'up' state immediately afterwards, irrespective of whether it was up or down before the repair.

An initiating event, i.e., one that demands the standby unit's services, occurs at random with probability β each period,

i.e., according to a Bernoulli trials process, so the times between events are independent and geometric. In this model we assume the operator is aware of those initiating events, to which the standby unit responded satisfactorily. This implies the unit was 'up' at that time, and although we neglect the time it was in operation, we say there is a (1-c) chance that its use will have caused it to go down by the end of the period. So if it was used at the n^{th} period after the unit was installed, there is a probability c, that it will be 'up' at the next period. (If c = 1, use is not hazardous.) If the standby system is down or is being inspected or repaired when an initiating event occurs, a catastrophic event occurs. The objective is to maximize the expected number of periods until a catastrophic event occurs.

The situation described can be treated as an infinite-state Markov decision process. The state space is describable as $S = \{(p,n),\ 0 \le p \le 1,\ n = 1,2,...\}$ where p is our belief that the unit is 'up' this period, and n is the number of periods since the standby unit was installed. There are three actions open to us at each state--do nothing, inspect or repair. Let V(p,n) be the maximum expected number of periods until a catastrophic event, given that this is the n^{th} period since installation, and p is our belief at this time that the unit is 'up'. Standard dynamic programming arguments [14] show that V(p,n) satisfies the optimality equation.

$$V(p,n) = \max\{W_1(p,n),\ W_2(p,n),\ W_3(p,n)\} \qquad (3.1)$$

where:

$$W_1(p,n) = 1 + (1-\beta)V(s_n p, n+1) + \beta p V(c, n+1)$$

$$W_2(p,n) = p[(1-(1-\beta)^M)/\beta + (1-\beta)^M V(i, n+M)]$$

$$+ (1-p)[(1-(1-\beta)^R)/\beta + (1-\beta)^R V(r, n+R)]$$

$$W_3(p,n) = (1-(1-\beta)^R)/\beta + (1-\beta)^R V(r, n+R)$$

Note that

$$(1-(1-\beta)^M)/\beta = \beta + 2\beta(1-\beta) + 3\beta(1-\beta)^2 + \ldots + M(1-\beta)^{M-1}$$

is the expected number of periods to pass, up to a maximum of M, until an initiating event occurs. $W_j(p,n)$ represents the payoff from an action; for example $W_1(p,n)$ corresponds to doing nothing, where with probability $(1-\beta)$ no demand occurs, while with probability βp an initiating event is successfully dealt with and with probability $(1-p)\beta$ a catastrophic event occurs. (3.1) is an example of Denardo's contraction operator approach to dynamic programming [4], and hence the optimal policy is independent of the past history of the system and consists of inspecting in state (p,n) if $W_2(p,n) > \max\{W_1(p,n), W_3(p,n)\}$ repairing if $W_3(p,n) > \max\{W_1(p,n), W_2(p,n)\}$, otherwise doing nothing.

As there is a probability $\beta(1 - \max_k\{s_k\})$ of a catastrophic event within two periods from any state and under any policy,

we have

$$1/\beta \leq V(p,n) \leq 2/\beta(1-\max_k(s_k)). \qquad (3.2)$$

It is easier to work with $\tilde{V}(p,n) = V(p,n) - 1/\beta$, which is the expected extra time until a catastrophic event because there is a standby unit. (3.1) then becomes

$$\tilde{V}(p,n) = \max\{\tilde{W}_1(p,n), \tilde{W}_2(p,n), \tilde{W}_3(p,n)\} \qquad (3.3)$$

where:

$$\tilde{W}_1(p,n) = p + (1-\beta)\tilde{V}(s_n p, n+1) + \beta p \tilde{V}(c,n+1)$$

$$\tilde{W}_2(p,n) = p(1-\beta)^M \tilde{V}(i,n+M) + (1-p)(1-\beta)^R \tilde{V}(r,n+R)$$

$$\tilde{W}_3(p,n) = (1-\beta)^R \tilde{V}(r,n+R) .$$

Lemma 3.1.

If s_n are non-increasing in n then $\tilde{V}(p,n)$ is convex and nondecreasing in p, and non-increasing in n.

<u>Proof</u>. Apply value iteration to solve (3.3); the iterates $V_m(p,n)$ satisfy

$$V_{m+1}(p,n) = \max \begin{cases} p + (1-\beta)\tilde{V}_m(s_n p, n+1) + \beta p \tilde{V}_m(c,n+1) \\ p(1-\beta)^M \tilde{V}_m(i,n+M) + (1-p)(1-\beta)^R \tilde{V}_m(r,n+R) \\ (1-\beta)^R \tilde{V}_m(r,n+R) . \end{cases} \qquad (3.4)$$

Let $\tilde{V}_0(p,n) = 0$ for all p and n, which is convex and non-decreasing in p and non-increasing in n. Since the sum of convex functions, and the maximum of convex functions is convex, if $\tilde{V}_m(p,n)$ is convex for all p and n so is $\tilde{V}_{m+1}(p,n)$. Thus by induction $\tilde{V}_m(p,n)$ is convex in p and since by [14], $\tilde{V}_m(.,.)$ converges to $\tilde{V}(.,.)$ the solution of (3.3), this limit function is also convex in p.

Again notice that if $\tilde{V}_m(p,n)$ is non-decreasing in p for all n, so is $p + (1-\beta)\tilde{V}_m(s_n p,n+1) + \beta p\tilde{V}_m(c,n+1)$ since $\tilde{V}_m(.,.) \geq 0$ and also $\max\{p(1-\beta)^M\tilde{V}_m(i,n+M) + (1-p)(1-\beta)^R\tilde{V}_m(r,n+R), (1-\beta)^R\tilde{V}_m(r,n+R)\}$ is non-decreasing in p. Hence $\tilde{V}_{m+1}(p,n)$, the maximum of these two non-decreasing functions, is non-decreasing and the induction step goes through. In the limit as $m \to \infty$ this proves $\tilde{V}(p,n)$ is non-decreasing in p.

For the dependence of $\tilde{V}(p,n)$ on n, we again use induction in the iterates $\tilde{V}_m(p,n)$: notice that (3.4) implies

$$\tilde{V}_{m+1}(p,n) - \tilde{V}_{m+1}(p,n+1) \geq$$

$$\max \begin{cases} \{(1-\beta)(\tilde{V}_m(s_n p,n+1) - \tilde{V}_m(s_{n+1}p,n+2)) + \beta p(\tilde{V}_m(c,n+1) - \tilde{V}_m(c,n+2)), \\ p(1-\beta)^M(\tilde{V}_m(i,n+M) - V_m(i,n+i+M)) + (1-p)(1-\beta)^R(\tilde{V}_m(r,n+R) \\ \qquad - \tilde{V}_m(r,n+1+R)), \\ (1-\beta)^R(\tilde{V}_m(r,n+R) - \tilde{V}_m(r,n+1+R))\}. \end{cases} \tag{3.5}$$

Assume $\tilde{V}_m(p,n) \geq \tilde{V}_m(p,n+1)$ for all p and n, then the fact $\tilde{V}_m(p,n)$ is non-decreasing in p means that, for all p,

$$\tilde{V}_m(s_n p,n+1) - \tilde{V}_m(s_{n+1}p,n+2) = (\tilde{V}_m(s_n p,n+1) - \tilde{V}_m(s_{n+1}p,n+1))$$

$$+ (\tilde{V}_m(s_{n+1}p,n+1) - \tilde{V}_m(s_{n+1}p,n+2)) \geq 0 . \qquad (3.6)$$

Hence (3.5) gives $\tilde{V}_{m+1}(p,n) - \tilde{V}_{m+1}(p,n+1) \geq 0$ for all p and n, and the induction hypothesis holds. Thus, the limit function $\tilde{V}(p,n)$ is also non-increasing in n.

These results help to describe the optimal policy.

Theorem 3.1

The optimal policy is given by a set of numbers p_n^*, $n = 1,2,\ldots$ where, n periods after installing the standby system, one <u>does nothing</u> in state (p,n) if $p > p_n^*$; inspects if $p \leq p_n^*$ and $(1-\beta)^M\tilde{V}(i,n) \geq (1-\beta)^R\tilde{V}(r,n)$; and repairs if $p \leq p_n^*$ and $(1-\beta)^M\tilde{V}(i,n) < (1-\beta)^R V(r,n)$. Notice if $i \geq r$, then one never repairs as $(1-\beta)^M\tilde{V}(i,n) > (1-\beta)^R\tilde{V}(r,n)$ for all n.

<u>Proof</u>. Notice that if $(1-\beta)^M\tilde{V}(i,n) \geq (1-\beta)^R\tilde{V}(r,n)$, then $\tilde{W}_2(p,n) > \tilde{W}_3(p,n)$ for all p; otherwise $\tilde{W}_3(p,n) \geq \tilde{W}_2(p,n)$. Now look at $\{p|\tilde{W}_1(p,n) \leq \max_{i=2,3}\{\tilde{W}_i(p,n)\}\}$, which is the set of states (p,n) where it is not best to do nothing. Since both $\tilde{W}_2(p,n)$ and $\tilde{W}_3(p,n)$ are linear in p and $\tilde{V}(p,n)$ is convex, we get for any p_1 and p_2 in the above region and any λ, $0 \leq \lambda \leq 1$.

$$\tilde{W}_i(\lambda p_1+(1-\lambda)p_2,n) = \lambda\tilde{W}_i(p_1,n)+(1-\lambda)\tilde{W}_i(p_2,n)$$

$$= \lambda\tilde{V}(p_1,n)+(1-\lambda)\tilde{V}(p_2,n) \geq \tilde{V}(\lambda p_1+(1-\lambda)p_2,n) \qquad (3.7)$$

where $i = 2$ or 3 depending on which is the maximum. Hence (3.7) implies $\max_{i=2,3} \tilde{W}_i(\lambda p_1+(1-\lambda p_2) \geq W_1(\lambda p_1+(1-\lambda)p_2)$ and so the region where it is not best to do nothing is convex.

From (2.3) we have

$$\tilde{V}(0,n) = \max\{(1-\beta)\tilde{V}(0,n+1), (1-\beta)^R V(r,n+R)\} . \qquad (3.8)$$

If it were best to do nothing at $p = 0$, this would imply $\tilde{V}(0,n) = (1-\beta)\tilde{V}(0,n+1)$, which contradicts $\tilde{V}(p,n)$ is non-increasing in n. Hence $(0,n)$ is in the convex region where it is not best to do nothing. Let p_n be the maximum value of p in this region and the result holds.

In fact the model can be rewritten so that the state space is countable, since not all possible values of p are possible. Let $S = \{(m,x,n), m = 0,1,2,..., x = i, r \text{ or } c, n = 1,2,3\}$ where (m,x,n) is the state when the unit is n periods since installation and m periods since the end of the last inspection, repair or successful response to an initiating event; $x = i$ if this last occurrence was an inspection that found it up; $x = r$ if it was a repair and $x = c$, if it was a successfully dealt with initiating event. The probability p that the unit is up in this state is $p(m,x,n) = x \prod_{k=1}^{m} s_{n-k}$ and so the optimality equation (3.3) becomes

$$\tilde{V}(m,x,n) = \max \begin{cases} p(m,x,n) + (1-\beta)\tilde{V}(m+1,x,n+1) \\ \quad + \beta p(m,x,n)\tilde{V}(0,c,n+1) \; ; \\ p(m,x,n)(1-\beta)^M\tilde{V}(0,i,n+M) \\ \quad + (1-p(m,x,n))(1-\beta)^R\tilde{V}(0,r,n+R) \; ; \\ (1-\beta)^R\tilde{V}(0,r,n+R) \end{cases} \tag{3.9}$$

and the optimal policy of Theorem 3.1 can be reinterpreted.

Corollary 3.1:

If, at n periods after installation, an initiating event is successfully dealt with, inspect or repair next in $T_c(n)$ periods unless there is another initiating event before then; if at n periods after installation, the unit has just been found to be 'up' on inspection, inspect or repair next in $T_i(n)$ periods unless an initiating event occurs; if at n periods after installation the unit has just finished a repair, then inspect or repair in $T_r(n)$ periods unless a prior initiating event occurs. If $i > r$ one always inspects, otherwise the repair or inspect decision depends on the number of periods since installation.

Proof. This is just a matter of pointing out that

$$T_c(n) = \min\{k \mid cs_n s_{n+1} \cdots s_{n+k} < p^*_{n+k}\} ,$$

$$T_i(n) = \min\{k \mid is_n s_{n+1} \cdots s_{n+k} < p^*_{n+k}\} ,$$

$$T_r(n) = \min_k\{k \mid rs_n s_{n+1} \cdots s_{n+k} < p^*_{n+k}\}.$$

Notice that $T_c(n)$, $T_i(n)$, $T_r(n)$ reflects the ordering of c, i and r, so if $c \geq i \geq r$ then $T_c(n) \geq T_i(n) \geq T|n)$, etc.

The dependence of this policy on n follows because the failure rate $(1-s_n)$ is age-dependent. We would expect that if s_n decreases with n, and consequently the failure rate is increasing, then $T_c(n)$, $T_i(n)$ and $T_r(n)$ will also be non-increasing in n. This reflects the fact that in the long run, the aging of the unit will lead to more frequent inspections. At the moment we are more interested in the effect of inspections and repair before aging starts to play a part. The interesting decision to replace an aging unit will not be analyzed at this time. From now on, assume that the failure rate is constant, which leads to the following simplification of Model 1.

Model 2

Assume $s_n = s$ for all n in Model 1, and $c = i$. This corresponds to thinking of an initiating event successfully dealt with as an inspection which takes zero time. The state space becomes $S = \{(m,x), m = 0,1,2, x = i, \text{ or } r\}$, the optimality equation (3.9) becomes

$$\tilde{V}(m,x) = \max \begin{cases} xs^m + (1-\beta)\tilde{V}(m+1,x) + \beta xs^m \tilde{V}(0,i) \; ; \\ xs^m(1-\beta)^M \tilde{V}(0,i) + (1-xs^m)(1-\beta)^R \tilde{V}(0,r); \\ (1-\beta)^R \tilde{V}(0,r) \; . \end{cases} \quad (3.10)$$

and the optimal policy is either of the form $\pi_i(T_i,T_r)$ or $\pi_r(T_i,T_r)$; $\pi_i(T_i,T_r)$ means inspect T_i periods after a successful response to an initiating event and T_i periods after the end of an inspection or T_r periods after the end of a repair, unless another initiating event occurs, whereupon inspect if T_i more periods elapse without another initiating event. $\pi_r(T_i,T_r)$ means repair T_i periods after a successfully-dealt-with initiating event, or T_r periods after last repair, unless another initiating event, or T_r periods after last repair, unless another initiating event occurs. Notice that one either always inspects or always repairs depending on the values of $(1-\beta)^M V(0,i)$ and $(1-\beta)^R V(0,r)$.

Although the state space is infinite we can apply variants of policy iteration and value iteration which solve the Markov decision process to find the optimal policy and optimal expected time to a catastrophic event. For any policy $\pi_i(T_i,T_r)$ there are only $T_i + T_r + 2$ states the unit can be in. So for, any expected policy we can calculate the corresponding expected time. Since the problem is equivalent to one with discount factor $(1-\beta(1-s))$, we can apply the bounds in White [22] to find a finite state approximation, whose value is within any prescribed amount of the optimal value. These bounds tell us how many states (m,x) we need to consider. The results given in Table 2 are the optimal policy and optimal expected time for different values of β, i, r, s, M and R, together with the expected times under other policies. The numbers we have chosen reflect an underlying model, in which inspections can be scheduled

at discrete times, say at multiples of a week. However, a repair or inspection takes only a fraction of this time. Although our theory was worked out for integer inspection and repair times, we take the same formula to approximate non-integer times. The inspection policy $\pi_i(1,0)$ means inspect one period after last inspection or last initiating event and immediately after a repair, while $\pi_r(0,100+)$ means repair immediately after any initiating event or at least 100 periods (100+) after a repair.

Notice the optimal policy is almost insensitive to whether $\beta = 0.05$ or 0.01 and the expected time to a catastrophic event is affected more by increases in i than r or even s. The policy $\pi_i(n,0)$ to inspect immediately after a repair is optimal if the probability of a repair not being effective is quite high, say 0.4. Similarly, the model suggests one should not inspect i.e., $\pi_r(.,.)$ if inspection is more hazardous than repair, $i < r$.

MODEL 3.

We might want to change our criterion from maximizing expected time until a catastrophic event to maximizing the probability that the system lasts at least n periods until a catastrophic event. This might be the case if the unit is to be completely replaced after n periods. If we apply this criterion to Model 2, $P_n(p)$ the probability that the system lasts at least n periods before a catastrophic event, given we believe it is 'up' at present with probability p, satisfies the

TABLE 2

β	i	r	s	M	R	OPTIMAL POLICY: ALWAYS INSPECT OR REPAIR	OPTIMAL POLICY: T_i	OPTIMAL POLICY: T_r	EXPECTED TIMES TO CATASTROPHIC EVENT: OPTIMAL V(0,i)	OPTIMAL V(0,r)	$\pi_i(1,1)$ V(0,i)	$\pi_i(1,1)$ V(0,r)	$\pi_i(4,4)$ V(0,i)	$\pi_i(4,4)$ V(0,r)	$\pi_i(12,12)$ V(0,i)	$\pi_i(12,12)$ V(0,r)
.1	.9	.9	.96	0.035	0.07	I	1	1	75.4	75.4	75.4	75.4	64.4	64.4	41.8	41.8
.1	.9	.95	.96	0.035	0.07	R	1	100+	97.4	98.0	78.8	79.2	67.4	68.5	42.8	44.1
.1	.9	.5	.96	0.035	0.07	I	2	0	71.6	70.9	48.6	45.6	43.5	36.2	35.1	26.0
.1	.5	.9	.96	0.035	0.07	R	0	100+	66.6	67.0	25.7	26.4	24.0	26.6	19.0	23.3
.1	.5	.5	.96	0.035	0.07	I	2	2	19.2	19.2	19.1	19.1	18.8	18.8	17.3	17.3
.1	.95	.9	.96	0.035	0.07	I	1	0	116.1	115.6	114.8	114.2	87.5	86.0	49.6	48.1
.1	.95	.9	.99	0.035	0.07	I	3	0	143.6	143.0	117.7	117.1	137.1	134.7	107.4	103.6
.1	.95	.9	.96	0.0175	0.035	I	1	0	146.1	145.8	141.0	140.2	90.4	88.9	49.8	48.3
.05	.95	.9	.96	0.035	0.07	I	1	0	229.4	228.9	216.5	215.9	157.9	156.3	78.4	76.7
.1	.9	.2	.99	0.035	0.07	I	4	0	79.1	76.9	32.4	26.3	44.0	25.3	54.1	22.5

optimality equation

$$P_0(p) = 1 \quad \text{for all } p.$$

$$P_{n+1}(p) = \max\begin{cases}(1-\beta)P_n(sp) + \beta p\, P_n(i)\\ p(1-\beta)^{\overline{M}}\, P_{n+1-\overline{M}}(i) + (1-p)(1-\beta)^{\overline{N}}P_{n+1-\overline{N}}(r)\\ (1-\beta)^{\overline{N}}\, P_{n+1-\overline{N}}(r)\end{cases} \tag{3.11}$$

where $\overline{M} = \min(M,n+1)$, $\overline{N} = \min(R,n+1)$. The optimal policy is again of a control-limit type.

Theorem 3.2.

The optimal policy to maximize the probability of lasting n periods is given by the sequence $p_1^*, p_2^*, \ldots p_n^*$, where with k periods to go, do nothing if $p > p_k^*$, inspect or repair if $p \le p_k^*$; repair if $(1-\beta)^{\overline{N}-\overline{M}}\, P_{n+1-\overline{N}}(r) \ge P_{n+1-\overline{M}}(i)$, and inspect otherwise.

Proof. As in Theorem 3.1, prove by induction that $P_n(p)$ is convex and non-decreasing in p and non-increasing in n. The convexity of $P_n(p)$ and the linearity of the second two terms in the maximization in (3.11) then gives the result.

If the state space is changed to $S = \{(m,x),\ m = 0,1,2,\ldots, x = i \text{ or } r\}$, by noting $p = xs^m$ at (m,x), the obvious change occurs in the optimal policy. In Table 3 we compare the maximum chance of lasting n periods before a catastrophic event for n = 10, 50 and 200 with the same chance under the policy π^* that maximizes the expected time to a catastrophic failure.

TABLE 3

β	i	r	s	M	R	OPTIMAL PROB. OF LASTING n PERIODS			PROB. OF LASTING n PERIODS UNDER π^*			I or R	OPTIMAL POLICY FOR EXPECTED VALUE π^*	
						n = 100	n = 500	n = 1000	n = 100	n = 500	n = 1000		T_i	T_r
.0035	.9	.9	.9986	1	2	.953	.777	.604	.952	.682	.515	I	40	40
.0035	.9	.5	.9986	1	2	.943	.761	.589	.941	.665	.501	I	46	0
.0035	.5	.9	.9986	1	2	.949	.709	.452	.946	.422	.361	R	0	100+
.0035	.5	.5	.9986	1	2	.832	.394	.155	.831	.150	.079	I	47	47
.0035	.95	.9	.9986	1	2	.965	.838	.710	.964	.791	.646	I	41	2
.0035	.95	.9	.9996	1	2	.973	.875	.773	.972	.856	.746	I	78	0

These figures are similar to those given for Model 2 except that the length of period is 1/10 of that there. So we can think of the probabilities as those of lasting 10, 50 or 100 weeks without a catastrophic failure. The optimal policy for maximizing expected time until failure does very well in almost all cases.

Model 4.

Suppose any information derived from having successfully dealt with initiating events, as in Model 2, were ignored; what changes would occur? We can no longer model this as a Markov decision process period by period since in these we cannot ignore information we know. However, we can construct a renewal theory model, for each end of inspection or end of repair is a type of renewal point. Thus we can define V_i, V_r as the maximum expected time to a catastrophic event starting immediately after a repair V_r or an inspection V_i. The rest of the model is the same as Model 2, with i, r, s, M, R having the same meaning as there. The optimality equation is then

$$V_i = \max_{T_i}\{L_i(T_i) + is^{T_i}((1-(1-\beta)^M)/\beta + (1-\beta)^M\ V_i) + p_i(T_i)((1-(1-\beta)^R/\beta) + (1-\beta)^R\ V_r)\}$$

$$V_r = \max_{T_r, W_r}\left\{\begin{array}{l} L_r(T_r) + rs^{T_r}((1-(1-\beta)^M)/\beta + (1-\beta)^M\ V_i) \\ \quad + p_r(T_r)((1-(1-\beta)^R)/\beta + (1-\beta)^R V_r); \\ L_r(W_r) + (rs^{W_r} + p_r(W_r))((1-(1-\beta)^R)/\beta \\ \qquad + (1-\beta)^R)V_r) \end{array}\right\} \tag{3.12}$$

where $L_p(T) = T - \sum_{i=0}^{T-2} [1-ps^i][1-(1-\beta)^{T-1-i}]$ is the expected number of periods, up to a maximum of T until a catastrophic event occurs, if p is the probability the unit is up at the start of the first period; and $p_x(T) = (1-xs^T) - \sum_{i=0}^{T-1} [\beta(1-\beta)^i][1-xs^{T-1-i}]$ is the probability that after T periods the unit is down but no catastrophic event has occurred given that initially it was up with probability x and down with probability 1-x. Again it is easier to work with $\tilde{V}_x = V_x - 1/\beta$ and the arguments of Markov renewal programming [7], show that the optimal policy is either $\pi_i(T_i,T_r)$, i.e., inspect T_i after last inspection and T_r after last repair, or $\pi_r(W_r)$, i.e., repair W_r after last repair. Using (3.12) we can calculate $\tilde{V}_i$, $\tilde{V}_r$ under these policies. For $\pi_i(T_i,T_r)$

$$V_r = \frac{r(1-s^{T_r})(1-(1-\beta)^M is^{T_i}) + i(1-s^{T_i})(1-\beta)^M rs^{T_r}}{(1-s)[(1-(1-\beta)^M is^{T_i})(1-(1-\beta)^R p_r(T_r)) - (1-\beta)^{M+R} p_i(T_i) rs^{T_r}]} ; \tag{3.13}$$

while under $\pi_r(W_r)$

$$V_r = \frac{r(1-s^{W_r})}{(1-s)(1-(1-\beta)^R(1-p_r(W_r)))} . \tag{3.14}$$

We calculate the optimal policy for the examples we did in Model 2, and so it is useful to compare the results with those given there. The results can be found in Table 4.

TABLE 4

						OPTIMAL POLICY MODEL 5				OPTIMAL POLICY MODEL 2			
β	i	r	s	M	R	REPAIR OR INSPECT	T_i	T_r (W_r)	V_r	REPAIR OR INSPECT	T_i	T_r	V(0,r)
.1	.9	.9	.96	0.035	0.07	I	2	1	75.8	I	1	1	75.4
.1	.9	.5	.96	0.035	0.07	I	3	1	53.0	I	2	0	70.9
.1	.5	.9	.96	0.035	0.07	R	-	(2)	69.2	R	0	100+	67.0
.1	.5	.5	.96	0.035	0.07	R	-	(100+)	22.4	I	2	2	19.2
.1	.95	.9	.96	0.035	0.07	I	3	1	112.7	I	1	0	115.6
.1	.95	.9	.99	0.035	0.07	I	5	1	148.4	I	3	0	143.0
.1	.9	.2	.99	0.035	0.07	I	100+	1	72.1	I	4	0	76.9

There are no great changes in the maximum time until a catastrophic event. Notice that there are examples where model 5 has a longer expected time. This may seem strange at first, since in Model 5, we are ignoring information--the occurrence of a successfully dealt with initiating event--which we use in Model 2. However to counterbalance this, in Model 5, it is implicit that after a successfully dealt with initiating event, the stand-by system is bound to be up, while in Model 2, it is only up with probability i. This also explains the difference in policy for the fourth example. Since repair and inspection are so bad, we do nothing to interfere with it under Model 5, but in Model 2 because after each successfully dealt with initiating event there is only a .5 chance it is up, we must keep inspecting it to see if this has occurred. Otherwise the only difference in policies is that the inspection intervals are slightly longer in Model 5 than in Model 2.

4. CONTINUOUS TIME MODEL WITH ONE UP STATE

In this section we look at the continuous time analogue of the standby unit model described in Section 3. Again, the standby unit can be either 'up' or 'down', and remains down either until it is inspected and repaired, or until a catastrophic initiating event occurs. An inspection takes a time of M, and if the unit works on inspection, nothing is done, and the lifetime of the unit thereafter is given by the distribution function $F_i(\cdot)$. The repair of a unit, found to be 'down' on inspection, takes, altogether with the inspection, a time of R and the lifetime distribution function thereafter is $F_r(\cdot)$. (The discrete time models have distribution functions corresponding to a point mass at zero together with a geometric distribution.) The times of the initiating events are given by a Poisson process with parameter ν, (so average inter-initiating event time is ν^{-1}). Again, we think of an initiating event that finds the unit up as the equivalent of an inspection. The problem is to find the times between inspections and between a repair and the next inspection which maximizes the expected time until a catastrophic event.

From the work of Doshi [5] on continuous time Markov decision processes, it follows that the optimal policy has a deterministic time T_i between inspections and a deterministic time T_r, between a repair and the next inspection. Moreover, if V_i, (V_r) are the maximum expected time to a catastrophic event starting after an inspection (repair), [5] implies V_i and V_r satisfy the optimality equation:

$$V_x = \sup_{T_x \geq 0}\Big\{ \int_0^{T_x} \nu e^{-\nu t}(t + \bar{F}_x(t)V_i)\,dt + T_x e^{-\nu T_x} + [e^{-\nu T_x}\,\bar{F}_x(T_x)$$

$$(\int_0^M t\nu e^{-\nu t}dt + Me^{-\nu M} + e^{-\nu M}\,V_i)] + e^{-\nu T_x}\,F_x(T_x)\,(\int_0^N t\nu e^{-\nu t}dt$$

$$+ Re^{-\nu R} + e^{-\nu R}V_r)\Big\} \qquad (4.1)$$

where $\bar{F}(t) = 1 - F(t)$ and $x = i$ or r. The T_i and T_r that actually maximize the R.H.S. of (4.1) are the optimal inspection times. Again, it is simpler to work with $\tilde{V}_x = V_x - 1/\nu$, which is the improvement in expected time until a catastrophic event when there is a standby system, over when there is no standby system. If $\tilde{V}_i(T_i,T_r)$, $\tilde{V}_r(T_i,T_r)$ are these improvements starting from an inspection and from a repair, when inter-inspection time is T_i and T_r is the time from repair to an inspection, we get by rearranging (4.1) that

$$\tilde{V}_x(T_i,T_r) = \int_0^{T_x} e^{-\nu t}\bar{F}_x(t)\,dt + \tilde{V}_i(T_i,T_r)\,[e^{-\nu T_x}e^{-\nu M}\,\bar{F}_x(T_x)$$

$$+ \int_0^{T_x} \nu e^{-\nu t}\,\bar{F}_x(t)\,dt] + \tilde{V}_r(T_i,T_r)\,e^{-\nu R}e^{-\nu T_x}F_x(T_x) \qquad (4.2)$$

Solving the system of equations (4.2) we get

$$\tilde{V}_i(T_i,T_r) = A(T_i,T_r)/C(T_i,T_r);\quad \tilde{V}_r(T_i,T_r) = B(T_i,T_r)/C(T_i,T_r) \qquad (4.3)$$

where

$$A(T_i,T_r) = (1-e^{-\nu(R+T_r)}\bar{F}_r(T_r)) \int_0^{T_i} e^{-\nu t}\bar{F}_i(t)\,dt$$

$$+ e^{-\nu(R+T_i)}\bar{F}_i(T_i) \int_0^{T_r} e^{-\nu t}\bar{F}_r(t)\,dt \,. \tag{4.4}$$

$$B(T_i,T_r) = (1-e^{-\nu(M+T_i)}\bar{F}_i(T_i)) \int_0^{T_r} e^{-\nu t}\bar{F}_r(t)\,dt$$

$$+ e^{-\nu(M+T_r)}\bar{F}_r(T_r) \int_0^{T_i} e^{-\nu t}\bar{F}_i(t)\,dt \,. \tag{4.5}$$

$$C(T_i,T_r) = 1 - e^{-\nu(M+T_i)}\bar{F}_i(T_i) - e^{-\nu(R+T_r)}F_r(T_r) +$$

$$e^{-\nu(M+R+T_i+T_r)}[F_r(T_r) - F_i(T_i)] - (1-e^{-\nu(R+T_r)}F_r(T_r)) \int_0^{T_i} \nu^{-\nu t}\bar{F}_i(t)\,dt$$

$$- e^{-\nu(T_i+R)}F_i(T_i) \int_0^{T_r} \nu e^{-\nu t}\bar{F}_r(t)\,dt \,. \tag{4.6}$$

If there are optimal finite inspection intervals T_i, T_r, they must satisfy for $x = i$ and r.

$$A_x'(T_i,T_r)/A(T_i,T_r) = B_x'(T_i,T_r)/B(T_i,T_r)$$

$$= C_x'(T_i,T_r)/C(T_i,T_r) \tag{4.7}$$

where

$$A_i' = \partial A/\partial T_i \quad \text{and} \quad A_r' = \partial A/\partial T_r, \text{ etc.}$$

In the special case where the extra time for a repair is zero and the lifetime of the unit is the same whether an inspection or a repair has just taken place, we can show that the optimal inspection times are finite. In this case $\tilde{V}_i = \tilde{V}_r = \tilde{V}$, $M = R$, $F_i(\cdot) = F_r(\cdot) = F(\cdot)$ and $T_i = T_r = T$, so (4.3) becomes

$$\tilde{V}(T) = A(T)/C(T) \tag{4.8}$$

where

$$A(T) = \int_0^T e^{-\nu t}\bar{F}(t)\,dt \tag{4.9}$$

$$C(T) = 1 - e^{-\nu(M+T)} - \int_0^T \nu e^{-\nu t}\bar{F}(t)\,dt \tag{4.10}$$

Lemma 4.1.

Optimal inspection time T^* is finite and $V(T^*) = \bar{F}(T^*)/\nu(e^{\nu M} - \bar{F}(T^*))$.

<u>Proof</u>

At a local maximum or minimum $\tilde{V}'(T) = 0$ which implies $h(T) = A'(T)C(T) - C'(T)A(T) = 0$ since $C(T)^2 > 0$, where

$$h(T) = e^{-\nu T}[\bar{F}(T)(1-e^{-\nu(M+T)}) - \nu e^{-\nu M}\int_0^T e^{-\nu t}\bar{F}(t)\,dt] \; ; \tag{4.11}$$

$h(0)$ is positive and though $h(\infty) = 0$ notice that $h(T) = e^{-\nu T}g(T)$ and as $T \to \infty$, $g(T) < 0$. This shows that $T = \infty$ is a minimum

turning point and that there is a finite turning point which is a maximum.

We could repeat the whole analysis for the continuous time analogue of the model where we ignore successfully-dealt-with initiating events, or at least do not consider them inspections. Using the notation of Model 1, the optimal values V_i and V_r satisfy

$$V_x = \sup_{T_x \geq 0}\{ \int_0^{T_x} t\,dt \int_0^t f_x(u)\nu e^{-\nu(t-u)}du + \overline{F}_x(T_x)[T_x + \int_0^M t\nu e^{-\nu t}dt$$

$$+ Me^{-\nu M} + e^{-\nu M}V_i] + (\int_0^{T_x} f_x(t)e^{-\nu(T_x-t)}dt)[T_x + \int_0^R t\nu e^{-\nu t}dt$$

$$+ Re^{-\nu R} + e^{-\nu R}V_r]. \tag{4.12}$$

The same analysis that led to (4.7) can be applied to (4.12) to find the optimal T_i and T_r. There is a difference in the special case when $M = R$, $F_i(\cdot) = F_r(\cdot) = F(\cdot)$, $T_i = T_r = T$ and $\tilde{V}_i = \tilde{V}_r = \tilde{V}$ where $\tilde{V} = V - 1/\nu$.

Now

$$\tilde{V}(T) = D(T)/K(T) \tag{4.13}$$

where

$$D(T) = \int_0^T \overline{F}(u)du \tag{4.14}$$

and

$$K(T) = 1 - e^{-\nu(M+T)}(1 + \nu \int_0^T \bar{F}(u)e^{\nu u}du) \qquad (4.15)$$

Lemma 4.2.

In this special case of Model 2, a sufficient condition for S*, the optimal inspection interval to be finite is that

$$r(\infty) > \frac{\nu}{1 + \nu\mu e^{-\nu M}} \qquad (4.16)$$

where $r(\infty) = \lim_{s\to\infty} f(s)/\bar{F}(s)$ and $\mu = \int_0^\infty tf(t)dt < \infty$ is the expected lifetime.

Proof.

At a local maximum or minimum of $\tilde{V}(T)$, $\tilde{V}'(T) = h(T)/K(T)^2 = 0$ where

$$h(T) = \bar{F}(T)(1-e^{-\nu(M+T)} - e^{-\nu(M+T)} \nu \int_0^T \bar{F}(u)e^{\nu u}du)$$

$$- (\int_0^T \bar{F}(u)du)\nu e^{-\nu(M+T)}(\int_0^T f(u)e^{\nu u}du + F(0))) \qquad (4.17)$$

Since $K(T)^2 > 0$, the condition $\tilde{V}'(T) = 0$ reduces to $h(T) = 0$. Notice that $h(0) = \bar{F}(0)[1-e^{-\nu M}] > 0$ but $h(\infty) = 0$. Thus to insure the maximum is not at $T = \infty$, we must show $h'(T)$ is positive as T tends to infinity. Differentiating h with respect to T, it follows that as T tends to infinity

$$h'(T) \to -r(\infty)(1 + \mu\nu e^{-\nu M}) + \nu^2\mu e^{-\nu M}(\nu b-1) + \frac{\mu\nu^2 e^{-\nu M}}{a} \qquad (4.18)$$

where

$$r(\infty) = \lim_{T\to\infty} f(T)/\overline{F}(T) = \lim_{T\to\infty} r(T), \quad a = \lim_{T\to\infty} e^{\nu T}\overline{F}(T) \quad ,$$

and

$$b = \lim_{T\to\infty} \left(\int_0^T \overline{F}(y)e^{\nu y}dy\right)/\overline{F}(T)e^{\nu T} \quad . \tag{4.19}$$

If $\overline{F}(T)e^{\nu T} = \exp(-\int_0^T (r(t)-\nu)dt) \to c$ as $T \to \infty$ then $b \to \infty$ and $h'(\infty)$ is positive; this certainly occurs if $r(\infty) > \nu$. If $\overline{F}(T)e^{\nu T} \to \infty$ as $T \to \infty$, then L'Hôpital's Rule says

$$b = \lim_{T\to\infty} \frac{\overline{F}(T)e^{\nu T}}{\nu\overline{F}(T)e^{\nu T} - f(T)e^{\nu T}} = \frac{1}{\nu - r(\infty)} \quad . \tag{4.20}$$

Thus

$$h'(T) \to -r(\infty)(1 + e^{-\nu M}\nu\mu) + \nu^2\mu e^{-\nu M} \frac{r(\infty)}{\nu - r(\infty)} \quad . \tag{4.21}$$

Since we are assuming $\overline{F}(T)e^{\nu T} \to \infty$ we have $r(\infty) \leq \nu$. If $r(\infty) < \nu$ then on checking when (4.21) is positive we get (4.16). Finally if $r(\infty) = \nu$, then $b = \infty$ and $h'(T)$ is still positive at $T = \infty$.

As an example suppose $\overline{F}(t) = we^{\lambda t}$, $t > 0$ so the unit has exponential lifetime with a probability 1-w of instantaneous failure, then the optimal inspection time T satisfies

$$e^{-(\nu+\lambda)T}[\lambda-\nu w)-(2\nu+\lambda)e^{-\nu M}e^{-\lambda T} + \nu(w+1)e^{-(\nu+\lambda)T} + \nu e^{-\nu M}e^{\lambda T}] = 0 \tag{4.22}$$

and the condition (4.16) that guarantees a finite solution to this equation is $\lambda > \nu(1 - we^{-\nu M})$.

5. TWO-UPSTATE MODEL

Model 7

We extend Model 2 of Section 3 to allow the unit to be in either one of two different up states: 1-up and 2-up, which have different failure rates. Let s_i, $i = 1,2$ be the probability of remaining in state i next period given that it is in state i this period, and $1-s_i$ is the probability it will fail in the next period. This model is intended to describe the situation in which a repair might only correct minor faults that caused the failure and not the underlying problem, which caused and will continue to cause these faults. We take as our state space $S = \{p,g) | 0 \leq p \leq 1, 0 \leq g \leq \infty\}$, where p is the belief that the unit is up, and g is the ratio of the probability the unit is in the 1-up state to the probability it is in the 2-up state. *Thus in the state (p,g) the belief the unit is* down, in the 1-up state and the 2-up state are respectively $1-p$, $gp/g+1$, $p/g+1$.

We assume that after a repair the unit is in state (r,w) and define $a = s_1/s_2$, where without loss of generality, we assume $s_1 \geq s_2$. The occurrence of a successfully-dealt-with initiating event is treated as an inspection which takes no time. Let $\tilde{V}(p,g)$ be the maximum extra number of periods under the best inspection policy until a catastrophic event, than if there was no standby unit (i.e., same definition as in Section 2). Again, Denardo's results [4] guarantee the optimal policy to be a deterministic one, it satisfies the optimality equation

$$\tilde{V}(p,g) = \max\{W_1(p,g),\ W_2(p,g),\ W_3(p,g)\} \tag{5.1}$$

$$W_1(p,g) = p + (1-\beta)\tilde{V}(s_2p(ag+1)/(g+1),ag) + \beta pV(i,g)$$

$$W_2(p,g) = p(1-\beta)^M\ \tilde{V}(i,g) + (1-p)(1-\beta)^R\ \tilde{V}(r,w)$$

$$W_3(p,g) = (1-\beta)^R\ \tilde{V}(r,w)\ .$$

The assumption is that an inspection affects the probability the unit is up, but not the ratio between the two up states, whereas a repair always returns the unit to the state (r,w). $(s_2p(ag+1)/g+1,ag)$ is the Bayesian updated belief of the state (p,g), using the fact that no initiating event occurred. The optimal policy for this model is given as follows.

Theroem 5.1.

The optimal policy is given by a function $p^*(g)$ and a number g^* so in state (p,g), it <u>does nothing</u> if $p > p^*(g)$, <u>inspects</u> if $p \leq p^*(g)$, $g > g^*$, and <u>repairs</u> if $p \leq p^*(g)$, $g \leq g^*$.

<u>Proof</u>

As in Theorem 3.1 an inductive proof on the iterates of value iteration proves that $\tilde{V}(p,g)$ is convex and non-decreasing in p and non-decreasing in g. Now define $W_g = \{p|\tilde{V}(p,g) > W_1(p,g)\}$; then the linearity of W_2 and W_3 and the convexity of $\tilde{V}$ in p guarantees W_g is convex, just as in Theorem 3.1. $\tilde{V}(0,g) = \tilde{V}(0,g')$ since if $p = 0$ there is only one state. From (5.1) it follows that

$$\tilde{V}(0,g) = \max\{(1-\beta)\tilde{V}(0,ag),\ (1-\beta)^N \tilde{V}(r,w) \quad . \tag{5.2}$$

By definition $\tilde{V}(r,w) \geq 0$ and if $\tilde{V}(0,g) = W_1(0,g) = (1-\beta)\tilde{V}(0,ag) = (1-\beta)\tilde{V}(0,g)$ then $\tilde{V}(0,g) = 0$, and hence $0 \in Wg$. Thus $Wg = [0,p^*(g)]$ and result holds, g^* satisfies $(1-\beta)^M V(i,g^*) = (1-\beta)^N V(r,w)$; and since $V(i,g)$ is non-decreasing in g this gives the division between inspection and repair.

Again we can rewrite the state space in terms of the number of periods since the last inspection and the last repair. Let $S = \{(m,n)\,|\,0 \leq m \leq n \leq \infty\}$ where (m,n) is the state which is m periods since the end of the last inspection or the end of repair if it followed from the last inspections and n non-inspection periods since the last repair. The state (m,n) is equivalent to $g = a^n w$,

$$p(m,n) = \begin{cases} s^m i(a^n w+1)/(a^{n-m}w+1) & \text{if} \quad n > m\ , \\ \\ s^m r(a^n w+1)/(a^{n-m}w+1) & \text{if} \quad n = m \end{cases} \tag{5.3}$$

If we define $p(m,n)$ according to (5.3), the optimality equation for this state space is

$$\tilde{V}(m,n) = \max\begin{cases} p(m,n) + (1-\beta)\tilde{V}(m+1,n+1) + \beta p(m,n)\tilde{V}(0,n) \\ \\ p(m,n)(1-\beta)^M \tilde{V}(0,N) + (1-p(m,n))(1-\beta)^R \tilde{V}(0,0) \\ \\ (1-\beta)^R \tilde{V}(0,0) \end{cases} \tag{5.4}$$

Theorem 5.1 can be reinterpreted for this state space.

Corollary 5.1.

The optimal policy is given by a function $m^*(n)$ and a number n^* so that at (m,n), do nothing if $m < m^*(n)$; inspect if $m \geq m^*(n)$, $n > n^*$; repair if $m \geq m^*(n)$, $n \leq n^*$. Notice if $i \geq r$, $n^* = 0$ and we always inspect.

Again we can use value iterations on a finite state approximation of the Markov decision model given by (5.4) (see White [22] for the bounds). This gives us the results found in Table 5. namely the optimal periods for inspections, counting from the last repair.

Note that the optimal inspection pattern appears to have short inter-inspection times just after a repair, which gradually *increase to long inspection times*, provided the system continues to be found up upon inspection. Hazardous inspection (i small) has a more drastic effect on the expected time to a catastrophic failure than similar changes in r, or s_1 and s_2.

Model 8

As in Section 3, we could also model the situation in which the information acquired from successfully-dealt-with initiating events is ignored. Then β, i, s_1, s_2, M, R are still defined as in Model 7, but immediately after an inspection or repair the time to the next inspection or repair is determined, and which kind it will be. Immediately after a repair suppose the unit has probability r_1, r_2 respectively of being in the 1-up or 2-up state. The decision points are immediately after a

TABLE 5

β	i	r	w	s_1	s_2	M	R	OPTIMAL POLICY OPTIMAL INSPECTION INTERVALS IN FIRST 50 PERIODS AFTER A REPAIR	V(0,0)
.1	.9	.9	1	.96	.9	0.035	0.07	1 (50 times)	75.0
.1	.9	.9	9	.96	.9	0.035	0.07	1 (50 times)	75.3
.1	.9	.5	1	.96	.9	0.035	0.07	1 (once) 2 (10 times) 3 (10 times)	45.4
.1	.5	.9	1	.96	.9	0.035	0.07	1 (repair)	64.2
.1	.5	.5	1	.96	.9	0.035	0.07	1 (20 times) 2 (15 times)	19.1
.1	.9	.9	.1	.96	.9	0.035	0.07	1 (10 times) 2 (20 times)	70.9
.1	.95	.9	9	.962	.1	0.035	0.07	2 (25 times)	112.9
.1	.95	.9	4	.967	.2	0.035	0.07	1 (3 times) 2 (24 times)	114.8
.1	.95	.9	2	.972	.5	0.035	0.07	1 (5 times) 2 (23 times)	115.2
.1	.95	.9	2	.975	.1	0.035	0.07	1 (2 times) 2 (24 times)	118.7
.1	.9	.9	1	.96	.5	0.035	0.07	1 (10 times) 2 (20 times)	74.6
.1	.9	.5	1	.96	.5	0.035	0.07	1 (2 times) 2 (once) 3 (15 times)	44.3
.1	.5	.9	1	.96	.5	0.035	0.07	1 (repair)	64.2
.1	.5	.5	1	.96	.5	0.035	0.07	1 (9 times) 2 (20 times)	19.1

repair, and immediately after an inspection, where it is important to know the number of operating periods n since the last repair. We denote the maximum expected times until a catastrophic event at these decision points as V_r, $V_{i,n}$ respectively. As in Model 4 we can write down the optimality equation connecting these values:

$$V_r = \max_{T_r, W_r} \left\{ \begin{array}{l} L(\underline{r},W_r) + (1-f(\underline{r},W_r))(((1-(1-\beta)^R)/\beta + (1-\beta)^R V_r); \\ L(\underline{r},T_r) + (r_1 s_1^{T_r} + r_2 s_2^{T_r})((1-(1-\beta)^M/\beta) + (1-\beta)^M V_{i,T_r}) \\ + (1-r_1 s_1^{T_r} - r_2 s_2^{T_r} - f(\underline{r},T_r))((1-(1-\beta)^R/\beta + (1-\beta)^R V_r) \end{array} \right\}$$

$$V_{i,n} = \max_{T_{i,n}} \left\{ \begin{array}{l} L(\underline{i}(n),T_{i,n}) + (\underline{i}(n)_1 s_1^{T_{i,n}} + \underline{i}(n)_2 s_2^{T_{i,n}})((1-(1-\beta)^M)/\beta \\ + (1-\beta)^M V_{i,n+T_{i,n}} + (1-\underline{i}(n)_1 s_1^{T_{i,n}} - \underline{i}(n)_2 s_2^{T_{i,n}} \\ - f(\underline{i}(n),T_{i,n})((1-(1-\beta)^R/\beta + (1-\beta)^R V_r) \end{array} \right\} \quad (5.5)$$

where $\underline{r} = (r_1, r_2, 1-r_1-r_2)$. If $\underline{p} = (p_1, p_2, p_3)$ where p_1 is the probability of being in the 1-up state, p_2 is the probability of being in the 2-up state, and p_3 is the probability of being down, then

$$L(\underline{p},T) = T - p_3 \sum_{k=1}^{T-1} (1-(1-\beta)^k) - p_1 \sum_{k=1}^{T-2} (1-(1-\beta)^{T-k-1})(1-s_1^k)$$

$$- p_2 \sum_{k=1}^{T-2} ((1-(1-\beta)^{T-k-1})(1-s_2^k) \quad (5.6)$$

is the expected time until a catastrophic failure in first T periods starting in state $\underline{p}$.

$$\underline{i}(n) = \left(\frac{ir_1 s_1^n}{r_1 s_1^n + r_2 s_2^n}, \frac{ir_2 s_2^n}{r_1 s_1^n + r_2 s_2^n}, 1-i\right)$$ is the state of the system

after inspection n operating periods after last repair, while

$$f(\underline{p},T) = (1-\beta)f(\underline{p},T-1) + p_1\beta(1-s_1^{T-1}) + p_2\beta(1-s_2^{T-1}) + \beta p_3 \quad (5.7)$$

is the probability there has been a catastrophic failure within T periods, starting in state p. Again, the general results of Markov renewal programming [7] show that the only possible optimal policies are $\pi_r(W)$, i.e., repair every W, or $\pi_i\{T_o, T_1, T_2, \ldots\}$ which is inspect T_0 periods after a repair, and T_k periods after the k^{th} inspection after a repair. In order to find the optimal policy it is easier to work with $\tilde{V}_x = V_x - 1/\beta$ again, and using (5.5) we can show that under the policy $\pi_r(W)$ if $\underline{r} = (r_1, r_2, 1-r_1-r_2)$

$$\tilde{V}_r = \left(\frac{r_1(1-s_1^W)}{(1-s_1)} + \frac{r_2(1-s_2^W)}{(1-s_2)}\right)/(1 - (1-f(\underline{r},W))(1-\beta)^R) \ . \quad (5.8)$$

Under the policy $\pi_i(T_0, T_1, \ldots)$ we get the following equations

$$\tilde{V}_r = \frac{r_1(1-s_1^{T_o})}{(1-s_1)} + \frac{r_2(1-s_2^{T_o})}{(1-s_2)} + (r_1 s_1^{T_o} + r_2 s_2^{T_o})[(1-\beta)^M \tilde{V}_{i,T_o}]$$

$$+ (1 - r_1 s_1^{T_o} - r_2 s_2^{T_o} - f(r,T_o))(1-\beta)^R \tilde{V}_r \ . \quad (5.9)$$

If $\tau_{k-1} = T_o + T_1 + \dots + T_{k-1}$

$$\tilde{V}_{i,\tau_{k-1}} = \frac{\underline{i}(\tau_{k-1})_1(1-s_1^{T_k})}{(1-s_1)} + \frac{\underline{i}(\tau_{k-1})_2(1-s_2^{T_k})}{(1-s_2)}$$

$$+ (\underline{i}(\tau_{k-1})_1 s_1^{T_k} + \underline{i}(\tau_{k-1})_2 s_2^{T_k})(1-\beta)^M \tilde{V}_{i,\tau_k}$$

$$+ (1-\underline{i}(\tau_{k-1})_1 s_1^{T_k} - \underline{i}(\tau_{k-1})_2 s_2^{T_k}$$

$$- f(\underline{i}(\tau_{k-1}),T_k)(1-\beta)^R \tilde{V}_r \qquad (5.10)$$

It appears somewhat difficult to solve (5.9) and (5.10) as we have an infinite set of equations. However, we can assume for all $\tau_k \geq N$, for some N, $\tilde{V}_{i,\tau_k}$ is approximately constant, since if a large number of periods have passed since the last repair, with no intervening failure, it is a good approximation to assume the unit is in the better of the two up states. This enables us to solve these equations using the bisection method reviewed in Thomas [18]. The method depends on the fact that if we substitute $\tilde{V}_r = c$ in the R.H.S. of (5.9) and (5.10) we can work back and solve for $\tilde{V}_r$ on the L.H.S. of (5.9). If c is the correct value of $\tilde{V}_r$, the L.H.S. of (5.9) is c, but if $c > \tilde{V}_r$, it follows easily that the L.H.S. of (5.9) will be greater than c, while if $c < \tilde{V}_r$ it will be smaller than c. Using this as the basis of the bisection method and taking all inspections more than 50 periods after a repair as the same, we get the forms of the approximately optimal policies found in Table 6; (the units of time are weeks).

TABLE 6

CASE	β	i	r_1	r_2	s_1	s_2	M	R	OPTIMAL INSPECTION INTERVALS AFTER A REPAIR	$\tilde{V}_r$
I	.1	.9	.6	.3	.96	.5	0.035	0.07	1 (5 times) 2 (23 times)	74.05
II	.1	.9	.333	.167	.96	.5	0.035	0.07	1 (1 time) 2 (1 time) 3 (1 time) 4 (11 times)	46.16
III	.1	.5	.6	.3	.96	.5	0.035	0.07	1 (repair)	62.10
IV	.1	.5	.333	.167	.96	.5	0.035	0.07	1 (once) 2 (once) 100+ thereafter	20.4

The parameters in the comparable continuous time model of Section 2 are (in units of weeks): $\nu = 01$, $\pi_1 = \frac{2}{3} = 1 - \pi_2$, $\delta_1 = .04$, $\delta_2 = 0.5$, $M = 0.035$ and $R = 0.07$. The corresponding best policies under the "short-long" inspection rule of Section 2 with inter-inspection times restricted to being multiplies of a week are as follows:

Table 7

Case	OKI	OKR	Best Policy	Best expected time to a catastrophic Event
I	0.9	0.9	1 (4 times), 2	61.09
II	0.9	0.5	1 (2 times), 3	42.19
III	0.5	0.9	3 (1, time), 1	29.37
IV	0.5	0.5	3 (1 time), ∞	19.98

The difference in policies for Case III results from the fact that the discrete time model allows a decision of repair without inspection. The differences in the policies for cases I, II, and IV come about because the continuous time model only allows inspection periods of two different lengths whereas the optimal policy in the discrete time model goes gradually from the length of the inspection period just after a repair to an asymptotic inspection period if the inspections are successful. However, subject to its restrictions, the policy of the continuous time model is comparable to that of the discrete time model.

The differences between the best expected times to a catastrophic event in the two models results from the discretization of time in Model 8. If the time interval in the discrete time model of Case I is taken to be 1/10 week instead of 1 week with the resulting change of parameters $\beta = .01$, $i = 0.9$, $r_1 = 0.6$, $r_2 = 0.3$, $s_1 = .996$, $s_2 = .95$, $M = 0.35$, $R = 0.7$, then the optimal policy is inspect 7 periods after a repair, and if up, then 8 periods later, then 9, 11, 13, 16, 18 and 20 periods and the expected time until catastrophic failure is 626.0 periods. In the original time scale this is a time of 62.6 weeks. Note that the difference between the expected time to a catastrophic event is now small for the two models. This suggests that the policy that was proposed in Section 2, while not optimal, is a good one.

6. CONCLUSIONS

The following conclusions can be drawn about the form of the optimal policy, by studying the models in this paper.

1) If the failure rate of the system increases with age, then the inspection intervals should decrease, and do. Numerical examples based on Model 1 have borne this out. The model calculations suggest optimal intervals based on the underlying parameters.

2) If there is only one state the unit can be in when it is 'up', and the probability of being up, i, is the same after each inspection and the probability of being up after a repair is also a constant r, then the optimal policy is to have one 'short' inspection interval after a repair, and a 'longer' inspection interval always thereafter (i > r) or else to repair at fixed intervals with no inspection (r considerably larger than i). The 'longer' inspection interval must always be at least as long as the 'short' initial inspection interval.

3) The results of 1) and 2) hold whether or not successfully-dealt with initiating events are considered as a type of inspection. However, there are considerable differences in the actual inspection periods for these two cases.

4) In order for the optimal inspection problem to require several 'short' inspection intervals followed by longer ones it is necessary to assume the unit can be in more than one 'up' state with different failure rates. In this case there is not an abrupt jump from 'short' inspection intervals to 'long', but a gradual increase in the inspection interval. However,

there is a suggestion that a policy comparable to the optimal one in which there is a sharp jump between short inspections and long ones, will give the expected time to a catastrophic event that is close to that achieved by the optimal policy.

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